The Dimension of an Amoeba

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ASGARD Math 2019

May 16, 2019
Basic Setup

- $X \subset (\mathbb{C}^*)^n$: irreducible subvariety.
- $\text{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$ by $(z_1, \ldots, z_n) \mapsto (\log |z_1|, \ldots, \log |z_n|)$.
- $\text{Ameoba } \mathcal{A}(X) := \text{Log}(X)$.
- Notion by Gelfand–Kapranov–Zelevinsky. Related to $A$-discriminants, real algebraic geometry, mirror symmetry, etc.
Example of Amoeba

\[ X = V(x + y + 1) \subset (\mathbb{C}^*)^2. \]

Figure 1.2 of *Tropical Algebraic Geometry* by Itenberg et al.
An amoeba has a canonical spine, which is the tropicalisation of $X$. 

$A(X)$ and $\text{Trop}(X)$. 

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**Question**

What about the (real) dimension of \( A(X) \)?
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What about the (real) dimension of $\mathcal{A}(X)$?

- Intuition/generic case: $\dim_\mathbb{R} X = 2 \dim_\mathbb{C} X$, and Log is “nice”, so $\dim_\mathbb{R} \mathcal{A}(X) = 2 \dim_\mathbb{C} X$. 
Well-known: \( \dim_{\mathbb{R}} \text{Trop}(X) = \dim_{\mathbb{C}} X \).

**Question**

What about the (real) dimension of \( A(X) \)?

- Intuition/generic case: \( \dim_{\mathbb{R}} X = 2 \dim_{\mathbb{C}} X \), and Log is “nice”, so \( \dim_{\mathbb{R}} A(X) = 2 \dim_{\mathbb{C}} X \).
- In general \( 2 \dim_{\mathbb{C}} X \) is an upper bound, but equality does not always hold.
Example (Hypersurface)

If $n > 2$ and $X$ is a hypersurface, then
$$\dim_{\mathbb{R}} \mathcal{A}(X) \leq \dim_{\mathbb{R}} \mathbb{R}^n = n < 2(n - 1) = 2 \dim_{\mathbb{C}} X.$$
Examples of Dimension Drop

Example (Hypersurface)
If \( n > 2 \) and \( X \) is a hypersurface, then
\[
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\]

Example (Torus)
\( X = \{(z^1 w^4, z^2 w^5, z^3 w^6) : z, w \in \mathbb{C}^*\} \) is a 2-dimensional subtorus. \( \mathcal{A}(X) = \text{span}\{(1, 2, 3), (4, 5, 6)\} \) is a 2-dimensional subspace. In general, the amoeba of a \( k \)-dimensional subtorus is a \( k \)-dimensional subspace.
More Examples of Dimension Drop

Example (Torus Action)

Suppose \( S \cdot X := \{(s_1z_1, \ldots, s_nz_n) : s \in S, z \in X\} = X \) for some \( k \)-dim torus \( S \).

\( X \mapsto X/S =: Y \subset (\mathbb{C}^*)^n/S \) (resp. \( \mathcal{A}(X) \mapsto \mathcal{A}(X)/\mathcal{A}(S) = \mathcal{A}(Y) \)) has fibers isomorphic to \( S \) (resp. \( \mathcal{A}(S) \)).

So \( \dim_{\mathbb{R}} \mathcal{A}(X) = k + \dim_{\mathbb{R}} \mathcal{A}(Y) \leq k + 2 \dim_{\mathbb{C}} Y = k + 2(\dim_{\mathbb{C}} X - k) = 2 \dim_{\mathbb{C}} X - k \).
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\[ X \mapsto X/S =: Y \subset (\mathbb{C}^*)^n/S \quad \text{(resp. } \mathcal{A}(X) \mapsto \mathcal{A}(X)/\mathcal{A}(S) = \mathcal{A}(Y)) \text{ has fibers isomorphic to } S \quad \text{(resp. } \mathcal{A}(S)). \]

So \( \dim_{\mathbb{R}} \mathcal{A}(X) = k + \dim_{\mathbb{R}} \mathcal{A}(Y) \leq k + 2 \dim_{\mathbb{C}} Y = k + 2(\dim_{\mathbb{C}} X - k) = 2 \dim_{\mathbb{C}} X - k. \)

- Nisse–Sottile (2018) suggested a program to understand amoebas better, including a conjecture about the dimension of amoebas.
Main Theorem

Theorem (Draisma–Rau–Y. 2018+)
\[ \dim_{\mathbb{R}} A(X) = \min \{ 2 \dim_{\mathbb{C}} X + 2 \dim_{\mathbb{C}} T - \dim_{\mathbb{C}} S \}, \text{ taken over } T \subset S \subset (\mathbb{C}^*)^n \text{ subtori such that } S \cdot T \cdot X = T \cdot X. \]
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Corollary

\[ \dim \mathcal{A}(X) = \min \{ 2 \dim \mathbb{C} S \cdot X - \dim \mathbb{C} S : S \subset (\mathbb{C}^*)^n \text{ subtorus} \}. \]
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**Theorem (Draisma–Rau–Y. 2018+)**

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**Corollary**

\[ \dim_{\mathbb{R}} A(X) = \min \{ 2 \dim_{\mathbb{C}} S \cdot X - \dim_{\mathbb{C}} S : S \subset (\mathbb{C}^*)^n \text{ subtorus} \}. \]

**Corollary (Conjecture of Nisse–Sottile)**

\[ \dim_{\mathbb{R}} A(X) < \min \{ 2 \dim_{\mathbb{C}} X, n \} \text{ iff } X \text{ admits a near/diminishing torus action.} \]
Theorem (DRY 2018+)

$$\dim_{\mathbb{R}} A(X) = \min \{ 2 \dim_{\mathbb{C}} X + 2 \dim_{\mathbb{C}} T - \dim_{\mathbb{C}} S : S \cdot \overline{T} \cdot X = \overline{T} \cdot X \}.$$
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Example (Trivial Bound)

Take \( T = S = \{1\} \). Then \( \dim_{\mathbb{R}} A(X) \leq 2 \dim_{\mathbb{C}} X \).
Verifications

**Theorem (DRY 2018+)**

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\]

**Example (Trivial Bound)**

Take \( T = S = \{1\} \). Then \( \dim_{\mathbb{R}} A(X) \leq 2 \dim_{\mathbb{C}} X \).

**Example (Hypersurface)**

Take \( T \) to be any generic 1-dim subtorus such that \( \overline{T} \cdot X = (\mathbb{C}^*)^n \). Then \( \dim_{\mathbb{R}} A(X) \leq 2 \dim_{\mathbb{C}} X + 2 \dim_{\mathbb{C}} T - \dim_{\mathbb{C}} (\mathbb{C}^*)^n = 2(n - 1) + 2 - n = n \).
Theorem (DRY 2018+)

\[ \dim_{\mathbb{R}} A(X) = \min \{ 2 \dim_{\mathbb{C}} X + 2 \dim_{\mathbb{C}} T - \dim_{\mathbb{C}} S : S \cdot \overline{T} \cdot X = \overline{T} \cdot X \}. \]

Proof of \( \leq \):

\[ \dim_{\mathbb{R}} A(X) \leq \dim_{\mathbb{R}} A(\overline{T} \cdot X) \leq 2 \dim_{\mathbb{C}} \overline{T} \cdot X - \dim_{\mathbb{C}} S \leq 2(\dim_{\mathbb{C}} X + \dim_{\mathbb{C}} T) - \dim_{\mathbb{C}} S. \]
Abs : $(\mathbb{C}^*)^n \to \mathbb{R}^n_0$ by $(z_1, \ldots, z_n) \mapsto (|z_1|, \ldots, |z_n|)$. 
$|X| := \text{Abs}(X)$ is the \textit{algebraic amoeba}, which is semi-algebraic.
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Goal: Find a \textit{rational} subspace $U$ of positive dimension that is contained in (almost) all $T_qA(X)$'s.  
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Idea: $\overline{|X|}$ is stable under the action of $R$, the real subtorus whose tangent space is $U$. $T, S$ will be inductively constructed using $R$ (and its complexification).
Goal: Find a rational subspace $U$ contained in (almost) all $T_q \mathcal{A}(X)$'s.

Lemma ("Swapping Quantifiers Principle")

"$\exists U, \forall q, U \leq T_q \mathcal{A}(X)$" is equivalent to "$\forall q, \exists U_q, U_q \leq T_q \mathcal{A}(X)$".

Proof: Suppose $|X| \approx \mathcal{A}(X)$ equals the union of (real-Zariski-closed) $\{ q : U \leq T_q \mathcal{A}(X) \}$ over all rational $U$'s. $|X|$ is irreducible and the union is countable, so one of such $\{ q : U \leq T_q \mathcal{A}(X) \}$'s is the whole of $|X|$.
Sketch of Proof: Swapping Real and Imaginary Parts

Since \( z = re^{i\theta} \), each \( T_zX \) decomposes into real and imaginary parts from 
\[
T_1(\mathbb{C}^*)^n = \mathbb{C}^n \cong \mathbb{R}^n \oplus i\mathbb{R}^n = T_1\mathbb{R}^n_0 \oplus T_0(S^1)^n.
\]

**Observation**

Abs takes the real part to \( T_{|z|}|X| \) and kills the imaginary part. But \( T_zX \) is a complex v.s., so its real part is precisely \( i \) times its imaginary part.
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Observation

Abs takes the real part to $T_{|z|}|X|$ and kills the imaginary part. But $T_zX$ is a complex v.s., so its real part is precisely $i$ times its imaginary part.

- Now it suffices to find $U$ from $Z_q := \text{Abs}^{-1}(q) \cap X \subset (S^1)^n$.
  (More precisely, from $\sum_{p \in Z_q} T_pZ_q$.)
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- Now it suffices to find $U$ from $Z_q := \text{Abs}^{-1}(q) \cap X \subset (S^1)^n$. (More precisely, from $\sum_{p \in Z_q} T_pZ_q$.)
- $U$ is essentially the tangent space of $\langle Z_q \rangle$. 
Corollary

$$\dim_{\mathbb{R}} A(X) = \min \{2 \dim_{\mathbb{R}} (S + \text{Trop}(X)) - \dim_{\mathbb{R}} S : S \leq \mathbb{R}^n \text{ rational} \}.$$
Corollary
\[ \dim_{\mathbb{R}} A(X) = \min \{ 2 \dim_{\mathbb{R}} (S + \text{Trop}(X)) - \dim_{\mathbb{R}} S : S \leq \mathbb{R}^n \text{ rational} \}. \]

Question
*Can \( \dim_{\mathbb{R}} A(X) \) be computed given \( \text{Trop}(X) \)?*
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- The above formula is computable (using real quantifier elimination) if the rationality condition is dropped.
Corollary
\[
\dim \mathcal{R} \mathcal{A}(X) = \min \{ 2 \dim \mathcal{R}(S + \text{Trop}(X)) - \dim \mathcal{R} S : S \leq \mathbb{R}^n \text{ rational} \}.
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Question

Can \( \dim \mathcal{R} \mathcal{A}(X) \) be computed given \( \text{Trop}(X) \)?

- The above formula is computable (using real quantifier elimination) if the rationality condition is dropped.
- But can we drop it?
Thank you!
Proposition (Nisse–Sottile)

\[ \dim_{\mathbb{R}} A(X) = \dim_{\mathbb{C}} X \text{ iff } X \text{ is a single torus orbit } S \cdot x. \]

Proof: \[ 2 \dim_{\mathbb{C}} S \cdot X - \dim_{\mathbb{C}} S = \dim_{\mathbb{C}} X \] for some subtorus \( S \). Since \( S \cdot X \supset X, S \cdot x \), we must have \( \dim_{\mathbb{C}} S \cdot X = \dim_{\mathbb{C}} S = \dim_{\mathbb{C}} X \), but this forces everything to be equal.