A DIRECT PROOF THAT NON-OVERLAPPING SIMPLICES FORM A TRIANGULATION FOR LAWRENCE POLYTOPES

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Abstract. We give a direct proof of (a strengthening of) Proposition 5.8 in Backman–Santos–Yuen: a non-overlapping collection of simplices of a Lawrence polytope that has the right cardinality is a triangulation of the polytope.

N.B. This manuscript is *not* self-contained at this moment. The reader is expected to refer to the original paper [2] and standard references such as [1, 4] for further background, terminology, and notation.

A *triangulation* of an oriented matroid is a collection of bases (geometrically, simplices) that satisfies the *pseudo-manifold property* (geometrically, the facets of the simplices align correctly) and the *non-overlapping property* (geometrically, the interior of the simplices do not intersect[1]). See [4, Chapter 2] for more equivalent definitions.

In [2], Backman, Santos, and myself studied several aspects of the *extension-lifting bijections*, including their connection with the triangulations of Lawrence polytopes; see also the work of Ding [3]. It is known that the bases in any triangulation of a Lawrence polytope are in bijection with the bases of the original oriented matroid [4, Proposition 4.12], in particular, all triangulations have the same cardinality. [2, Proposition 5.8] states a converse of this fact: any non-overlapping collection of bases constructed from the bases of the original oriented matroid forms a triangulation.

The proof given in [2] uses a non-trivial result of Santos [4, Theorem 4.14], while we remarked that a direct proof is possible (which in turns gives a new proof of several statements in the literature), and the purpose of this manuscript is to describe such a proof. In fact, we prove a slightly stronger statement where we only assume the cardinality and the non-overlapping property of the collection.

Proposition 1. *Let M be an oriented matroid and let $\Lambda(M)$ be its the Lawrence polytope. Then any non-overlapping collection of $|\mathcal{B}(M)|$ bases of $\Lambda(M)$ form a triangulation of $\Lambda(M)$.*

We adopt the notation in [2]: M is on the ground set E whereas $\Lambda(M)$ is on $E \sqcup \overline{E}$, we identify a subset of M with the same subset of $E \subset \Lambda(M)$ and a signed subset (X^+, X^-) of M with the subset $X^+ \cup \overline{X^-}$ of $\Lambda(M)$. For elements of $\Lambda(M)$, $\iota : \Lambda(M) \to \Lambda(M)$, $\underline{\cdot} : \Lambda(M) \to M$ are the involution map $e \leftrightarrow \overline{e}$ and projection map $e, \overline{e} \mapsto e$, respectively.

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[1]As Santos pointed out, this geometric interpretation is valid only when the pseudo-manifold property is present, so the interpretation is more of an intuition rather than a rigorous statement here.

Proof. Fix a collection $\mathcal{T}$ of bases as in the proposition statement. Recall that every basis of $\Lambda(M)$ is of the form $B^A := B \cup \overline{B} \cup A \cup \overline{E \setminus (A \cup B)}$, where B is a basis of M and $A \subset E \setminus B$.

We first prove that $\mathcal{T}$ does not contain two bases B^{A_1}, B^{A_2} of $\Lambda(M)$ coming from the same basis B of M. Suppose not, by swapping the role of A_1, A_2 if needed, we may assume there exists $a \in A_1 \setminus A_2$. Pick the signed circuit $C = (C^+, C^-)$ of M supported on the fundamental circuit of a with respect to B such that $a \in C^+$. Then B^{A_1}, B^{A_2} overlap on the signed circuit $(C^+ \cup \overline{C^-}, \overline{C^+} \cup C^-)$ of $\Lambda(M)$.

By considering cardinality, there is a bijection between the bases of M and the bases in $\mathcal{T}$ via $B \mapsto B^{A_B} =: \widetilde{B}$; this is the assumption in the original [2, Proposition 5.8]. By [2, Lemma 5.7], $\mathcal{T}$ is induced by some *circuit signature* σ of M: σ is a map that picks, for every circuit $\underline{C}$ of (the underlying matroid of) M, a signed circuit $\sigma(\underline{C})$ of M supported on $\underline{C}$; moreover, whenever $\underline{C}$ is the fundamental circuit with respect to some basis B of M, $\sigma(\underline{C})$ is contained in $\widetilde{B} \in \mathcal{T}$. It remains to show that $\mathcal{T}$ satisfies the pseudo-manifold property. Let $\widetilde{B}$ be a basis in $\mathcal{T}$ coming from $B \in \mathcal{B}(M)$, and let e be an element of $\widetilde{B}$, we verify that $\widetilde{B} \setminus e$ is either contained in another basis in $\mathcal{T}$ or its complement contains a positive cocircuit.

Case I: $e \notin B \cup \overline{B}$.
The complement of $\widetilde{B} \setminus e$ now contains the positive cocircuit $\{e, \iota(e)\}$ of $\Lambda(M)$.

Case II: $e \in B \cup \overline{B}$.
Let $\underline{D}$ be the fundamental cocircuit of $\underline{e}$ with respect to B, and let D be the signed cocircuit of M supported on $\underline{D}$ such that $e \in D$ (when D is viewed as a subset of $\Lambda(M)$). We are done if the complement of $\widetilde{B} \setminus e$ contains the positive circuit $D = D^+ \cup \overline{D^-}$ of $\Lambda(M)$, so we assume $(\widetilde{B} \setminus e) \cap D = \{e_1, \dots, e_t\}$ is non-empty.

For each e_i, denote the basis $(B \setminus \underline{e}) \cup \{\underline{e_i}\}$ of M by B_i (the fact that B_i is a basis follows from that $\underline{e_i}$'s are elements in the fundamental cocircuit of $\underline{e}$ with respect to B), and the fundamental circuit of $\underline{e_i}$ with respect to B by $\underline{C_i}$. $\underline{C_i}$ and $\underline{D}$ are both fundamental circuit and cocircuit of B and B_i, respectively, so $\underline{C_i} \cap \underline{D} = \{\underline{e}, \underline{e_i}\}$. By the choice of D we know that $e_i \in D$, while $\sigma(\underline{C_i}) \subset \widetilde{B}$ implies that $e_i \in \sigma(\underline{C_i})$ as well. Therefore by the orthogonality between signed circuits and signed cocircuits, we have $\iota(e) \in \sigma(\underline{C_i}) \subset \widetilde{B_i}$, and $e \notin \widetilde{B_i}$.

Next we note that if there exists some $f \in \widetilde{B} \setminus \widetilde{B_i}$ besides e, then $\underline{f} \in \underline{D}$: we still have $\{f, \iota(f)\} \subset \widetilde{B_i}$ for any $\underline{f} \in B \setminus \underline{e}$, while for $\underline{f} \in E \setminus (B \cup \underline{D})$, the fundamental circuits of $\underline{f}$ with respect to B and B_i are the same, so the same signed circuit picked out by σ is present in both $\widetilde{B}$ and $\widetilde{B_i}$. Let $C_f = (C_f^+, C_f^-)$ be the signed circuit of M supported on the fundamental circuit of $\underline{f}$ with respect to B such that $f \in C_f$ when viewed as a subset of $\Lambda(M)$. If $f \notin D$, then by the orthogonality between D and C_f, we have $e \in C_f$, and $\widetilde{B}, \widetilde{B_i}$ overlap on the signed circuit $(C_f^+ \cup \overline{C_f^-}, C_f^- \cup \overline{C_f^+})$ of $\Lambda(M)$ (with the element "a" in the definition of non-overlapping being either f or $\iota(f)$, depending on which side f is in), a contradiction. Hence, we must have $f = e_j$ for some $j \neq i$.

Now we build a digraph Γ whose vertices are $e_1, \dots, e_t$ by including a directed edge $e_i \to e_j$ whenever $e_j \in \widetilde{B} \setminus \widetilde{B_i}$. We claim that Γ is a tournament. For each pair of i, j, let $\underline{C_{i,j}}$ be the fundamental circuit of $\underline{e_j}$ with respect to B_i, which is also the fundamental circuit of $\underline{e_i}$ with respect to B_j. The same $\sigma(\underline{C_{i,j}})$ is contained in both $\widetilde{B_i}$ and $\widetilde{B_j}$, and by the orthogonality between $\sigma(\underline{C_{i,j}})$ and D, exactly one of $e_j \in \widetilde{B} \setminus \widetilde{B_i}$ and $e_i \in \widetilde{B} \setminus \widetilde{B_j}$ happen.

Case II-a: Γ is acyclic.
It is well-known that Γ must have a sink e_i in this case, so $\widetilde{B} \setminus \widetilde{B_i} = \{e\}$, i.e., $\widetilde{B} \setminus e$ is contained in $\mathcal{T} \ni \widetilde{B_i} = (\widetilde{B} \setminus e) \cup \{\iota(e_i)\}$.

Case II-b: Γ is not acyclic.
It is in turn well-known that Γ must contain a directed 3-cycle $e_i \to e_j \to e_k \to e_i$. Let $\underline{C}$ be the fundamental circuit of $\underline{e_k}$ with respect to B_i. The absence of the directed edge $e_i \to e_k$ in Γ means $e_k \in \widetilde{B_i}$, and since $C = (C^+, C^-) := \sigma(\underline{C}) \subset \widetilde{B_i}$, we have $e_k \in C$, and $\iota(e_i) \in C$ by the orthogonality between C and D. On the other hand, the directed edge $e_j \to e_k$ means $e_k \in \widetilde{B} \setminus \widetilde{B_j}$, so we have $\iota(e_k) \in \widetilde{B_j}$ instead. Moreover, the absence of $e_j \to e_i$ implies $e_i \in \widetilde{B_j}$, so we must have $\iota(C) \subset \widetilde{B_j}$, a contradiction as now $\widetilde{B_i}, \widetilde{B_j}$ overlap on the signed circuit $(C^+ \cup \overline{C^-}, \overline{C^+} \cup C^-)$ of $\Lambda(M)$ (with the element "a" being either e_k or $\iota(e_k)$). □

References

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