The Critical Group of an Adinkra

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Outline

- Introduction to Adinkras
- Introduction to Critical Groups and Signed Graphs
- The Laplacian and Critical Group of an Adinkra
- Open Problems and Future Directions

Some Physics (that I don't really understand)

Supersymmetry (SUSY): Every boson ϕ has an associated fermion ψ and vice versa.

Physicists are interested in *SUSY superalgebras*, some particularly interesting/useful ones satisfy:

- the algebra is generated by $Q_1, \ldots, Q_N, H := \sqrt{-1}\partial_t$;
- the generators act on $\{\phi_1, \ldots, \phi_m; \psi_1, \ldots, \psi_m\}$;
- each Q_i takes a boson to some fermion up to signs and H, vice versa;
- $Q_iQ_j + Q_jQ_i = 2\delta_{ij}H$ and $Q_iH = HQ_i$.

If we pretend H does nothing, we get a *Clifford algebra* $Cl(0, N)$: $Q_i^2 = I, Q_iQ_j = -Q_jQ_i$.

Adinkras

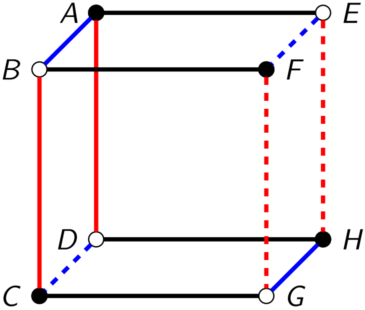
Definition (Faux–Gates 2004)

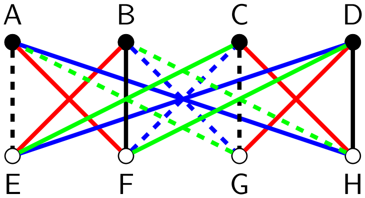
An *Adinkra/Cliffordinkra* is a (connected, simple) graph with each edge colored by one of N colors and is either solid or dashed, such that:

1. the graph is bipartite;
2. every vertex is incident to exactly one edge of each color;
3. for each pair of distinct colors, the graph restricted to these colored edges is a disjoint union of 4-cycles;
4. each bi-colored 4-cycle contains an odd number of dashed edges.

Related to: Error correcting codes, cubical cohomology, combinatorial maps and Riemann surfaces, etc.

Examples

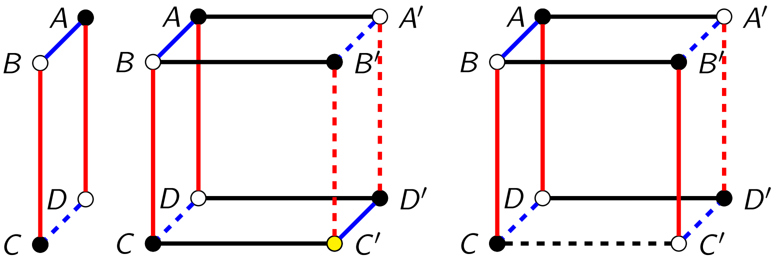




Some Operations on Adinkras

Definition

- Prism: Create a new copy with solid/dashed reversed, match old and new vertices with solid edges of a new color.
- Vertex Switch: Reverse solid/dashed for edges incident to a vertex.



Classification of the Underlying Graphs

Definition

A *doubly even code* $\mathcal{C}$ is a subspace of $\mathbb{F}_2^N$ such that the weight of every element is a multiple of 4.

EXAMPLE: The row space of $\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$.

Theorem (DFGHILM 2008)

A graph admits an Adinkra structure iff it is a hypercube modulo some doubly even code, i.e., the vertex set is $\mathbb{F}_2^N/\mathcal{C}$, *and* $[x] \sim [y]$ *whenever* $x \sim y$ *in the hypercube.*

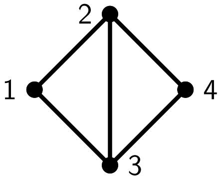
EXAMPLE: $K_{4,4}$ is the quotient of the 4-dim cube by the code $\langle(1\ 1\ 1\ 1)\rangle$.

Laplacian, Critical Group, and Matrix–Tree Theorem

Definition

- Laplacian: $L := D - A$, D is the diagonal matrix whose entries are the vertex degrees, A is the adjacency matrix.
- Critical group: $\operatorname{coker} L := \mathbb{Z}^V / \operatorname{row}_{\mathbb{Z}} L = K(G) \oplus \mathbb{Z}$.

G :



$$, L = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}, K(G) \cong \mathbb{Z}/8\mathbb{Z}$$

As known as *sandpile group* or *Jacobian*, and is related to *chip-firing* and many areas.

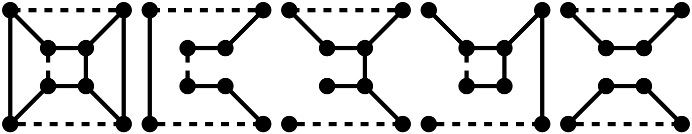
Theorem (Kirchhoff's Matrix–Tree Theorem)

$|K(G)| = \#$ *of spanning trees of* G.

Signed Graphs

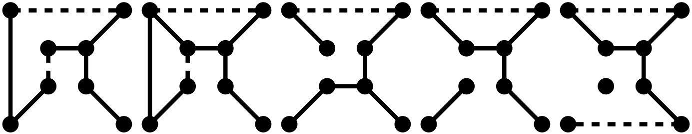
Definition

- Signed Graph: A graph with a signing $E \to \{+,-\}$ of the edges.
- Spanning Tree: Every component has exactly one cycle, which must have an odd number of $-ve$ edges.
- Weight of a ST: $w(T) := 4^{\#\text{ of components}}$.



Weights: 4,4,4,16

Non-examples:

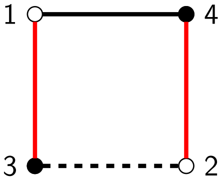


Laplacians and Matrix–Tree Theorem of Signed Graphs

Definition

- Laplacian: $L := D - A$, but an entry of A is -1 if the edge is $-ve$.
- Critical group: $K(G) := \operatorname{coker} L$.

G :



$$, L = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & 1 & -1 \\ -1 & 1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}, K(G) \cong (\mathbb{Z}/2\mathbb{Z})^2$$

Theorem (Zaslavsky 1982)

$|K(G)| = \det L = \sum_T w(T)$.

A Quadratic Relation of L

L is the signed Laplacian of an Adinkra of N colors.

Proposition

$L^2 - 2NL + (N^2 - N)I = 0.$

Proof: We show the simpler relation $A^2 = NI$. It is well-known that the (u, v)-entry of A^2 is the weighted count of length 2 paths from u to v. When $u = v$, the paths are $u \to w \to u$, so the sum is $\deg(u) = N$.
When $u \neq v$, every path $u \xrightarrow{i} w \xrightarrow{j} v$ is paired with a unique path $u \xrightarrow{j} w' \xrightarrow{i} v$ of opposite sign, so the sum is 0.

Spectrum of L

Corollary

The eigenvalues of L are $N \pm \sqrt{N}$.

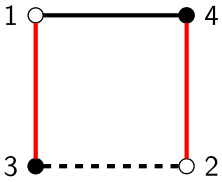
Proposition

Each eigenvalue has multiplicity $\#V/2$, hence $\det L = (N^2 - N)^{\#V/2}$.

REMARK: The matrix theory of Adinkras is thus similar to that of *strongly regular graphs*.

Colored Laplacians

Replace each entry of an edge of color i by an indeterminate $\pm x_i$, and each diagonal entry by $\sigma := \sum x_i$.



$$\mathcal{L} = \begin{pmatrix} x_1 + x_2 & 0 & -x_1 & -x_2 \\ 0 & x_1 + x_2 & x_2 & -x_1 \\ -x_1 & x_2 & x_1 + x_2 & 0 \\ -x_2 & -x_1 & 0 & x_1 + x_2 \end{pmatrix}$$

- $\mathcal{L}^2 - 2\sigma\mathcal{L} + (\sigma^2 - \rho)I = 0$, where $\rho := \sum x_i^2$.
- $\det \mathcal{L} = (\sigma^2 - \rho)^{\#V/2} = (\sum_{i \neq j} x_i x_j)^{\#V/2}$.

The Critical Group of an Adinkra

Notation: $K_A := \text{coker } L \cong \oplus \mathbb{Z}/f_i\mathbb{Z}$ with $f_1|\ldots|f_{\#V}$, $\#V = 2^{N-\dim \mathcal{C}} := 2^{N-k}$.

N	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
1	t: $(1,0)$				
2	t^2: $(1^2, 2^2)$				
3	t^3: $(1^4, 6^4)$				
4	t^4: $(1^8, 12^8)$	d_4: $(1^2, 2^2, 6^2, 12^2)$			
5	t^5: $(1^{16}, 20^{16})$	$d_4 \oplus t$: $(1^8, 20^8)$			
6	t^6: $(1^{32}, 30^{32})$	$d_4 \oplus t^2$: $(1^{16}, 30^{16})$	d_6: $(1^8, 30^8)$		
7	t^7: $(1^{64}, 42^{64})$	$d_4 \oplus t^3$: $(1^{32}, 42^{32})$	$d_6 \oplus t$: $(1^{16}, 42^{16})$	e_7: $(1^8, 42^8)$	
8	t^8: $(1^{128}, 56^{128})$	$d_4 \oplus t^4$: $(1^{64}, 56^{64})$	$d_6 \oplus t^2$: $(1^{32}, 56^{32})$	$e_7 \oplus t$: $(1^{16}, 56^{16})$	
		h_8: $(1^{56}, 2^8, 28^8, 56^{56})$	$d_4 \oplus d_4$: $(1^{24}, 2^8, 28^8, 56^{24})$	d_8: $(1^8, 2^8, 28^8, 56^8)$	e_8: $(1^2, 2^6, 28^6, 56^2)$

Theorem (IKKY 2021+)

Let N' be the largest odd number dividing $N^2 - N$. Then the odd component of K_A is $(\mathbb{Z}/N'\mathbb{Z})^{\#V/2}$.

- In "most" cases, K_A is just $(\mathbb{Z}/(N^2 - N)\mathbb{Z})^{\#V/2}$.
- But the nice formula doesn't hold for an infinite family of Adinkras (e.g. for type D codes with $4|N$).

Upper Bound of p-rank via Eigenvalues

Theorem (Lorenzini 2008)

Let $M \in \mathbb{Z}^{n \times n}$ and let $\lambda_1, \ldots, \lambda_t$ be the distinct nonzero eigenvalues of M. Then every nonzero invariant factor of M divides $\prod \lambda_i$.

Corollary

For an Adinkra, every invariant factor divides $N^2 - N$. Hence for each $p | N^2 - N$, at least $\#V/2$ invariant factors are divisible by p.

The p-Sylow Subgroup for $p|N-1$

Theorem (Elementary Divisor Theorem)

Let $M \in \mathbb{Z}^{n\times n}$ and let g_i be the GCD of all $i \times i$ minors of M. Then $f_1 \dots f_i = g_i, \forall i$.

Corollary

For each prime $p|N-1$, let p^α be the largest power of p dividing $N-1$. Then the p-Sylow subgroup of K_A is $(\mathbb{Z}/p^\alpha\mathbb{Z})^{\#V/2}$.

PROOF: The $\#V/2 \times \#V/2$ submatrix NI has determinant $N^{\#V/2}$, which is relatively prime to p, so the first $\#V/2$ invariant factors must also be relatively prime to p. This forces each remaining invariant factor to be divisible by p^α.

Some Algebraic Setup

Set $x_1 = x, x_2 = \ldots = x_N = 1$ in the colored Laplacian $\mathcal{L}$ to obtain $\hat{L}$.
Note that $\det \hat{L} = (2(N-1)x + (N-1)(N-2))^{\#V/2}$.

We can further modulo the entries by p and/or setting $x = 1$.

$$\begin{array}{ccc} \mathbb{Z}[x] & \xrightarrow{x \mapsto 1} & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{F}_p[x] & \xrightarrow{x \mapsto 1} & \mathbb{F}_p \end{array}, \quad \begin{array}{ccc} \hat{L} & \longrightarrow & L \\ \downarrow & & \downarrow \\ \tilde{L} & \longrightarrow & \overline{L} \end{array}$$

Lemma

$\#$ of invariant factors of L divisible by p
$=$ corank of $\overline{L}$
$=$ $\#$ of invariant factors of $\tilde{L}$ divisible by $x-1$.

Lower Bound of p-rank for odd $p|N$

Since $\det \tilde{L} = (-2(x-1))^{\#V/2}$, there can be at most $\#V/2$ invariant factors of $\tilde{L}$ divisible by $x-1$.

Corollary

For each odd $p|N$, exactly $\#V/2$ invariant factors of L are divisible by p, and the p-Sylow subgroup of K_A is $(\mathbb{Z}/p^\alpha\mathbb{Z})^{\#V/2}$.

Proposition

Let $M \in \mathbb{Z}^{n\times n}$ and let p be a prime. Then the $\#$ of invariant factors of M divisible by p equals

$$\min\{\mathrm{ord}_{x-1} \det \hat{M} \in \mathbb{F}_p[x] : \hat{M} \in \mathbb{Z}[x]^{n\times n}, \hat{M}|_{x=1} = M\}.$$

Shape of Invariant Factors

Proposition

$f_i f_{\#V-i+1} = N^2 - N$, and $f_{2i-1} = f_{2i}, \forall i$.

PROOF IDEA: Use the block structure of L to produce the invariant factors of L from those of the top-right block.

Corollary

Lorenzini's bound is tight for Adinkras.

Theorem (Hung–Y. 2021+)

If a graph or signed graph has two distinct nonzero Laplacian eigenvalues, and is not $K_{m,m}$ or $K_{1,p}$ up to switching, then Lorenzini's bound is tight.

Special Nice Cases

Observation

If $N \not\equiv 0 \pmod 4$, *then* $K_A \cong (\mathbb{Z}/(N^2 - N)\mathbb{Z})^{\#V/2}$.

Observation

If A is a prism, then $K_A \cong (\mathbb{Z}/(N^2 - N)\mathbb{Z})^{\#V/2}$.

Proposition

If $1 \notin \mathcal{C}$, *then* $K_A \cong (\mathbb{Z}/(N^2 - N)\mathbb{Z})^{\#V/2}$.

Proof Idea: Follows from the results on Cayley graphs of $\mathbb{F}_2^r$ by Gao–Marx-Kuo–McDonald (2019+).

Applications

Corollary (Bai 2003, also a former conjecture of Reiner)

The (ordinary) critical group of a hypercube has exactly $2^{N-1} - 1$ non-trivial ($\neq 0, 1$) invariant factors.

Proof Idea: The key part is to find the number of even invariant factors, but it equals that of a cubical Adinkra's as we forget signs over $\mathbb{F}_2$.

Corollary (Special case of conjectures of Gao–Marx-Kuo–McDonald)

If $\ker(M)$ *is a doubly even code, then the number of even nonzero invariant factors of* $K(\mathrm{Cayley}(\mathbb{F}_2^N, M))$

- *is at least* $2^{N-1} - 1$. [Conjecture 6.1];
- *is odd* [Conjecture 6.2 (assuming certain eigenvalue hypothesis)].

Future Directions and Open Problems

When $K_A \ncong (\mathbb{Z}/(N^2 - N)\mathbb{Z})^{\#V/2}$:

- WHAT IS THE 2-SYLOW SUBGROUP (OR 2-RANK)?
- Does changing the dashing affect the critical group?

When $K_A \cong (\mathbb{Z}/(N^2 - N)\mathbb{Z})^{\#V/2}$:

- Does a Smith Normal Form of $\hat{L}$ exist? (Condition is necessary.)

General:

- Interpretation of the results via representation theory?
- Other interesting instances of the "lift to polynomial ring" trick?

Thank you!