The Combinatorics of Break Divisors

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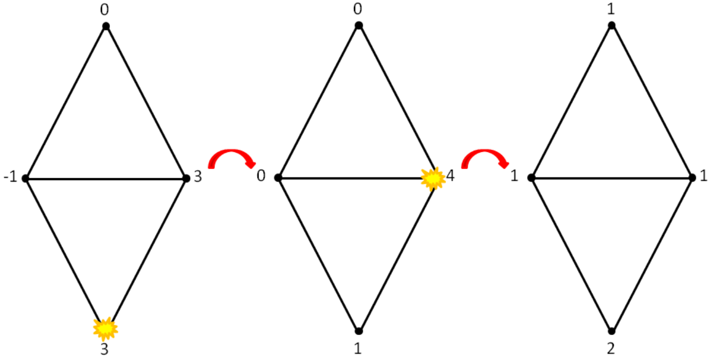
Divisors and Chip-Firing on Finite Graphs

Definition

- *Divisor*: A configuration to put "chips" on vertices.
- $\mathrm{Div}^d(G)$: The set of degree d divisors.
- *Chip-firing*: Remove $\deg(v)$ chips from v, and send one chip along each incident edge.



Jacobian

Definition

The *Jacobian* $\text{Jac}(G) = \text{Pic}^0(G)$ is $\text{Div}^0(G)/\sim$.
$D \sim D'$ if D, D' differ by a series of chip-firing moves.
$\text{Pic}^d(G) := \text{Div}^d(G)/\sim$.

- Also known as: (degree 0) Picard group, sandpile group, critical group, the group of components...
- Related to MANY areas: statistical physics/probability, combinatorics, graph theory, commutative algebra, representation theory...
- Tropical geometry: $\text{Jac}(G)$ keeps track of the information of degenerated line bundles over semistable models.

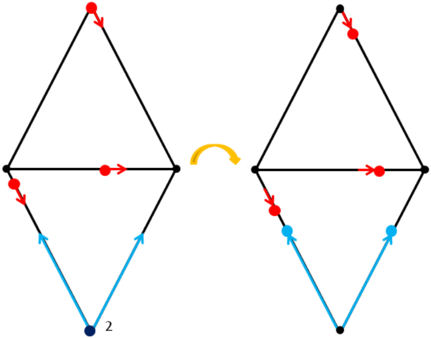
Theorem (Kirchhoff's Matrix–Tree Theorem (Version N+1))

$|\text{Jac}(G)|$ *equals the number of spanning trees of* G.

Metric Graphs and Divisors

Definition

- *Metric graph* Γ: Metric version of a graph, each edge is an interval (as metric space).
- *Divisor*: We can now add chips inside edges.
- *Chip-firing*: Fire chips across cuts.



Tropical Jacobian

$\mathrm{Pic}^0(\Gamma)$ has the canonical structure of a g-dimensional real torus ($g := |E| - |V| + 1$ is the *genus* of Γ).

1. Fix a model G for Γ, and work with $C_1(G; \mathbb{R}) \cong \mathbb{R}^E$.
2. The subspace $H_1(G; \mathbb{R})$ and lattice $H_1(G; \mathbb{Z})$ are spanned by cycles of G (over $\mathbb{R}$ and $\mathbb{Z}$, respectively).
3. The *tropical Jacobian* $\mathrm{Jac}(\Gamma)$ is the torus $H_1(G; \mathbb{R})/H_1(G; \mathbb{Z})$.

- $\mathrm{Pic}^0(G) \cong \mathrm{Jac}(\Gamma)$: Given a divisor $D = p_1 + \ldots + p_k - q_1 - \ldots - q_k$, pick a path γ_i from p_i to q_i, map D to $\gamma_1 + \ldots + \gamma_k \in C_1(G; \mathbb{R})$, project to $H_1(G; \mathbb{R})$ and then to $\mathrm{Jac}(\Gamma)$.

The Geometry of $\mathrm{Div}_+^g(\Gamma) \to \mathrm{Jac}(\Gamma)$

$\mathrm{Div}_+^g(\Gamma) \cong \Gamma^g/\mathfrak{S}_g$: The space of degree g *effective* divisors.

Theorem (Mikhalkin–Zharkov 2008, An–Baker–Kuperberg–Shokrieh 2014)

There exists a unique continuous section to the map $\mathrm{Div}_+^g(\Gamma) \to \mathrm{Jac}(\Gamma)$.

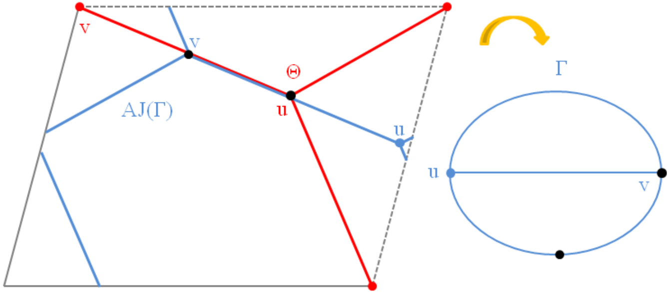
Remark

Not true in the classical world, e.g., when C is a genus 2 hyperelliptic curve, $\mathrm{Div}_+^2(C) = \mathrm{Sym}^2(C) \to \mathrm{Jac}(C)$ *is a blow-down map.*

Proof of Existence (Mikhalkin–Zharkov)

Idea: Construct the section via tropical intersection theory.

1. Map Γ to $\mathrm{Jac}(\Gamma)$ by the *Abel-Jacobi map* AJ.
2. Take the intersection of $AJ(\Gamma)$ and a shifted copy of $\Theta = \sum_{i=1}^{g-1} AJ(\Gamma) \subset \mathrm{Jac}(\Gamma)$ (the *tropical theta divisor*). (Θ can also be defined using tropical theta function.)
3. Tropical magic: even when the intersection is not transverse, one can still take stable intersection.

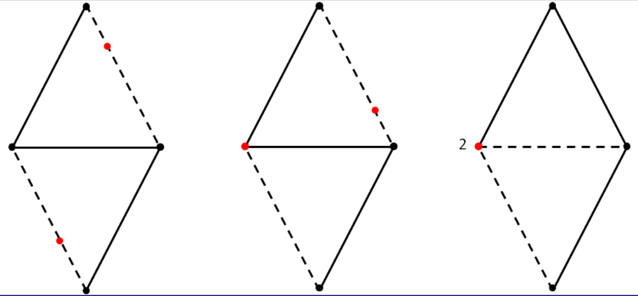


Proof of Existence (ABKS)

Idea: Write down the section combinatorially.

Definition

A *break divisor* of Γ is a divisor obtained by first choosing a spanning tree T of G and then put a chip at every edge not in T.



Theorem (ABKS 2014)

Break divisors form a set of representatives for $\mathrm{Pic}^g(\Gamma)$ *(and* $\mathrm{Pic}^g(G)$*).*

The ABKS Decomposition

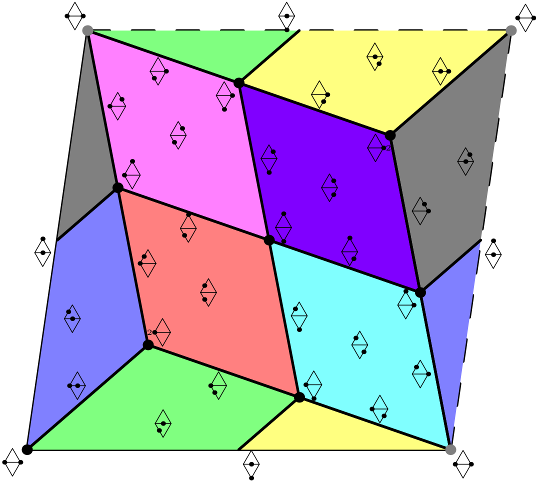
Observation

- *For a spanning tree T with $e_1, \dots, e_g \notin T$, the break divisors associated to T (essentially) form a parallelotope $\mathcal{C}_T := e_1 \times \dots \times e_g$.*
- *Vertices of the cells are the elements of* $\mathrm{Jac}(G)$.

Corollary (ABKS 2014)

$\mathrm{Jac}(\Gamma) = \bigcup_T \mathcal{C}_T$, *and the interior of two distinct cells are disjoint.*

ABKS Decomposition: The Picture



Theorem (Kirchhoff's Matrix–Tree Theorem (Version N+2))

The volume of Jac(Γ) *is the determinant of the reduced Laplacian.*

Applications in Algebraic Geometry

Proposition (Baker 2017+)

The Zhang measure (tropical canonical measure) of Γ is the distribution of chips of a random break divisor.
Analogous discretized statement.

Theorem (Shen 2017+)

For a nodal curve X with dual graph Γ, any Simpson compactification *of* $\text{Pic}^g(X)$ *has a stratification "anti-isomorphic" to the face poset of the ABKS decomposition of* $\text{Jac}(\Gamma)$. *Specifically,*

1. $\widehat{\text{Pic}^g(X)} = \coprod_F J^F$, *where F runs over the faces of the ABKS decomposition.*
2. $\mathrm{codim}(J^F) = \dim(F), \forall F$.
3. *J^F is "glued" to $J^{F'}$ iff F' is a face of F.*

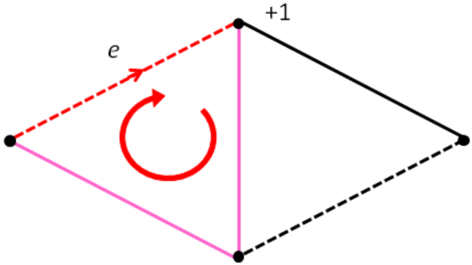
Geometric Bijections

Choose some generic vector v. Shift each vertex along $-\epsilon \mathrm{v}$ so it lands in a cell, now we have a bijection Φ_{v} between spanning trees and $\mathrm{Jac}(G)$.

- Setup: v picks an orientation for every cycle C of G, by choosing the one with positive inner product with v.

Theorem (Y. 2017 (Matrix–Tree Theorem Version N+3))

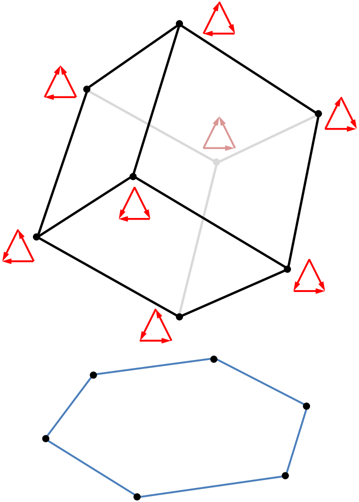
Φ_{v} maps a spanning tree T to a break divisor as follows: orient each $e \notin T$ according to its fundmental cycle, put a chip at the head of e.



Zonotopes

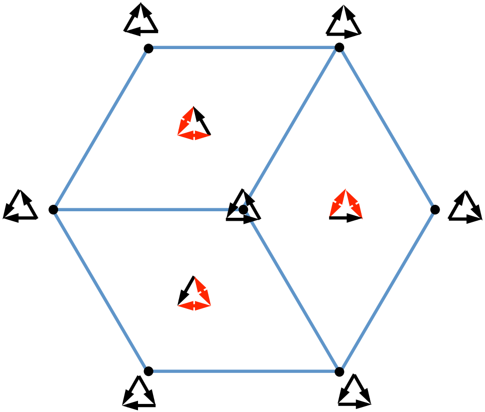
A: The incidence matrix of a graph G (or any matrix).

The *zonotope* Z_A is the projection of $[0,1]^E$ by A.



Zonotopal Tilings

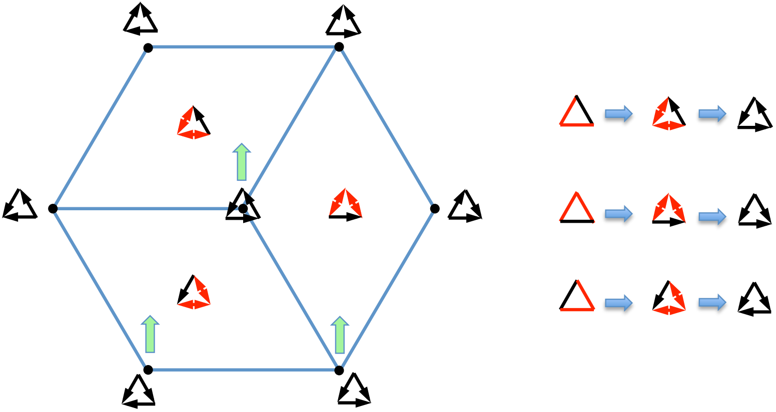
Using a variation of regular subdivision, Z_A can be partitioned into parallelotopes (with respect to some vector w).



To experts: By Bohne–Dress, Zonotopal tilings ⇔ Single-element liftings.

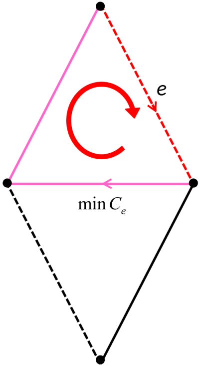
Shifting (Again)

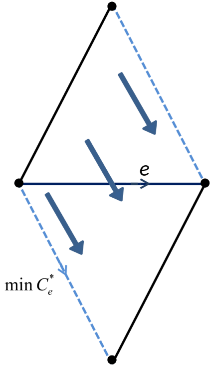
Shift the vertices along some generic v as last time. Again we have a bijection $\Phi_{w,v}$ between spanning trees and (some) vertices.
Vertices $\approx \text{Pic}^m(G) \approx \text{Jac}(G)$ by taking indegree sequence.



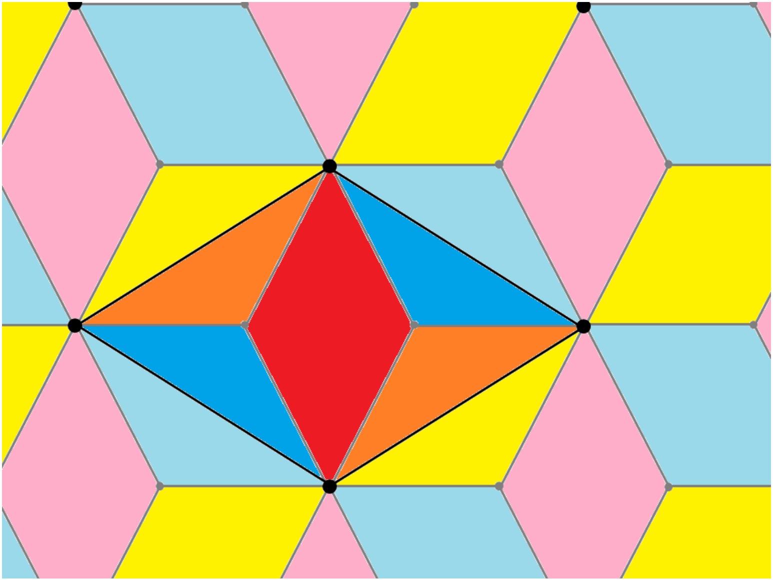
The vectors w (for tiling) and v (for shifting) pick an orientation for every cycle and cut of G. The bijection $\Phi_{w,v}$ can be described as follows:

- Given a spanning tree T.
- Orient each $e \notin T$ according to its fundmental cycle.
  Orient each $e \in T$ according to its fundmental cut.
- (Optional) Put a chip at the head of every edge.





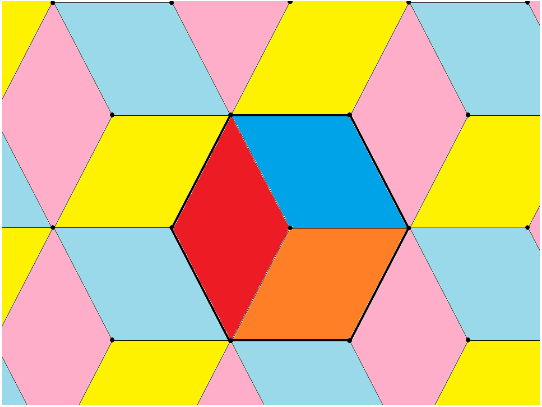
Tiling of Jac(Γ)



Tiling of Zonotopes

Proposition (Y. 2017+)

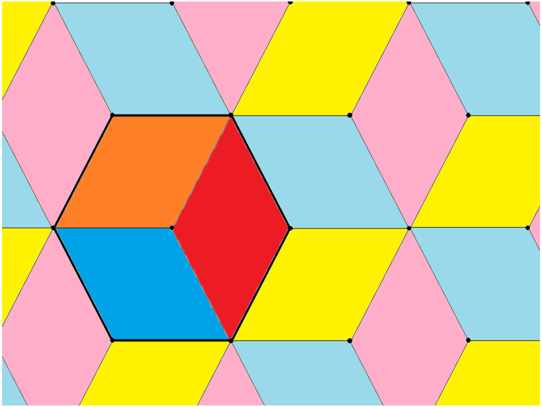
The ABKS decomposition of G (lifted to the universal cover), coincides with certain zonotopal tiling of $Z_{M(G)^*}$ (tiling the whole space).



Tiling of Zonotopes

Observation

Many zonotopal tilings lead to the same ABKS decomposition.



Group Action–Tiling Duality

Observation

Any bijection ϕ between Jac(G) *and the collection of spanning trees induces a* Jac(G)*-action (torsor) by* $[D] \cdot T := \phi([D] + \phi^{-1}(T))$.

Proposition (Y. 2017+)

Two "geometric" group actions for a plane graph G are isomorphic iff the corresponding tilings for its **dual graph** *G^* differ only by a translation. General statement works in the setting of* regular matroids.

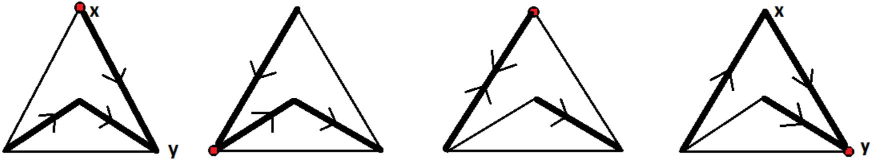
- Technical Statement: $\Phi_{\mathrm{w,v}}$ and $\Phi_{\mathrm{w,v'}}$ induce isomorphic group actions iff the extension tilings of G^* induced by v and v′ differ by a translation.
- Punchline: The group action "dual" to ABKS decomposition was studied by combinatorists before.

Special Group Actions for Plane Graphs

Olivier Bernardi's process (08): Fix a starting edge (v, f) in a plane graph.

- For every spanning tree T, starting with (v, f), walk along edges in T.
- Cut every $e \notin T$ twice, put a chip at the end that was being cut first.

There is an analogous *rotor-routing* that also produces $\mathrm{Jac}(G)$-actions.



Theorem (Baker–Wang 2017, Chan–Church–Grochow 2015)

All Bernardi bijections and rotor-routings induce isomorphic group actions.

Proposition (Y. 2017+)

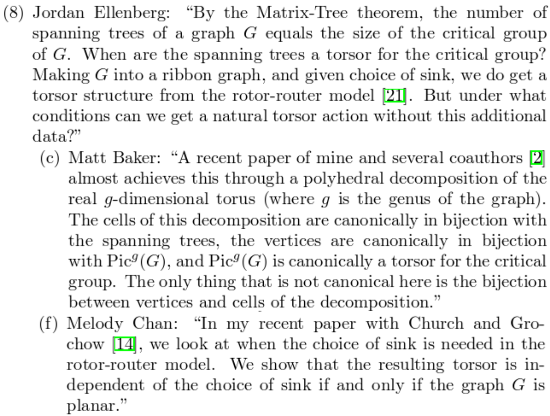
The ABKS decomposition of G is the "dual" of such action for G^*.

Some History

The canonical group action for a plane graph

"$\Leftrightarrow$"

The canonical tropical structure on its dual


(8) Jordan Ellenberg: "By the Matrix-Tree theorem, the number of spanning trees of a graph G equals the size of the critical group of G. When are the spanning trees a torsor for the critical group? Making G into a ribbon graph, and given choice of sink, we do get a torsor structure from the rotor-router model [21]. But under what conditions can we get a natural torsor action without this additional data?"

(c) Matt Baker: "A recent paper of mine and several coauthors [2] almost achieves this through a polyhedral decomposition of the real g-dimensional torus (where g is the genus of the graph). The cells of this decomposition are canonically in bijection with the spanning trees, the vertices are canonically in bijection with $\mathrm{Pic}^g(G)$, and $\mathrm{Pic}^g(G)$ is canonically a torsor for the critical group. The only thing that is not canonical here is the bijection between vertices and cells of the decomposition."

(f) Melody Chan: "In my recent paper with Church and Grochow [14], we look at when the choice of sink is needed in the rotor-router model. We show that the resulting torsor is independent of the choice of sink if and only if the graph G is planar."

From the problem session of the 2013 AIM workshop on chip-firing (scribed by Sam Hopkins).

Thank you!