Filtrations of Tope Spaces of Oriented Matroids

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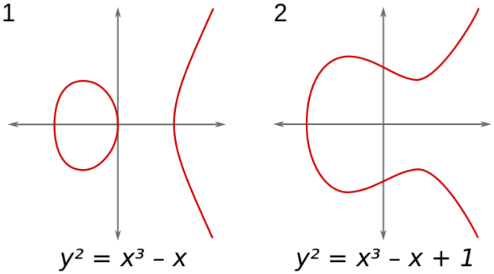
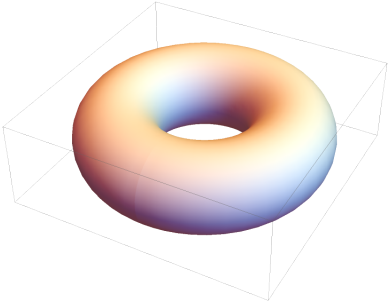
Topological motivation I

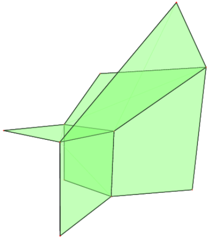
What can we learn about the topology of a real algebraic variety X from its complex counterpart?





Topological motivation II

What can we learn about the topology of an algebraic variety X from its tropicalization $\operatorname{trop} X$?



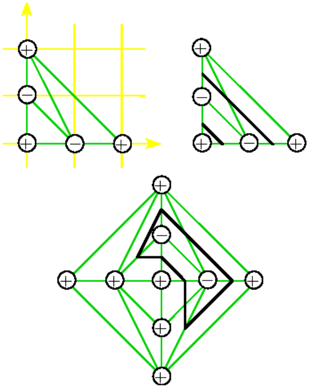
[Itenberg–Katzarkov–Mikhalkin–Zharkov '19]: Remember extra data on each cone and do sheaf (co)homology.

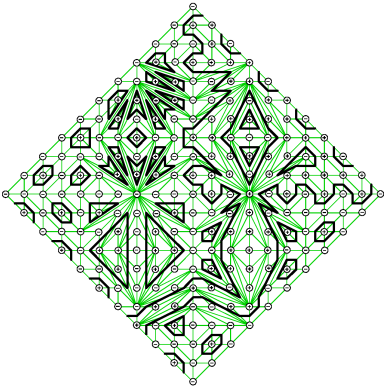
Example (Zharkov '13)

The tropical cohomology of a matroid fan B_M is the *Orlik–Solomon algebra* $\mathrm{OS}^*(M)$ of M.

When the two motivations collide

Viro's Patchworking (1980's): constructing real algebraic hypersurfaces (up to isotopy) with tropical data plus sign data.

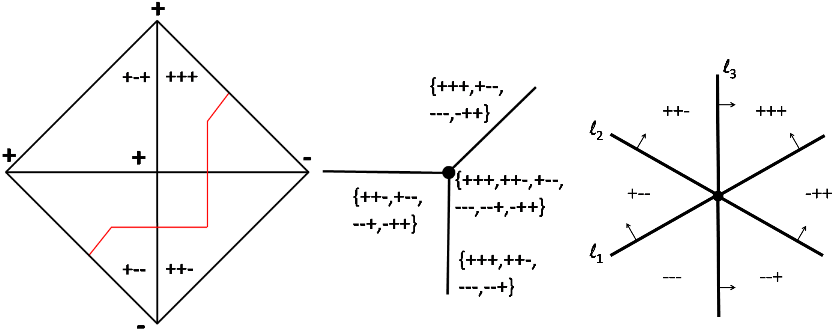




Given a regular triangulation of $n\triangle_{d-1}$ and signs assigned at the vertices. Take the "zero locus" within each cell, and glue all loci together.

From Patchworking to Oriented Matroids

- Instead of unfolding the tropical locus, we can just remember which orthants $\subset \{+,-\}^d$ each face is in: this is a *sign cosheaf*.



- Matroid fan + sign data = oriented matroid. [Rau–Renaudineau–Shaw '22]

Slogan: OMs $\cong$ the local theory of real (smooth) tropical geometry.

The Work of Renaudineau–Shaw

Theorem (Renaudineau–Shaw '23 (Conjecture of Itenberg '05))

Let $\mathbb{R}X \subset \mathbb{P}^{n+1}$ be a real algebraic hypersurface constructed from patchworking. Then

$$b_q(\mathbb{R}X) \leq \begin{cases} h^{q,n-q}(\mathbb{C}X) + 1, & q \neq n/2 \\ h^{q,q}(\mathbb{C}X), & q = n/2 \end{cases}.$$

PROOF IDEA:

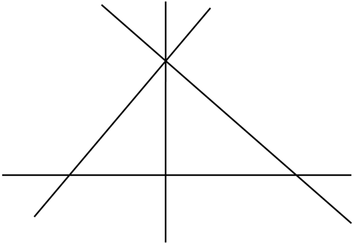
- The topology of $\mathbb{R}X$ from the sign cosheaf homology of trop X;
- The topology of $\mathbb{C}X$ from defining a filtration on the sign cosheaf and relating it with tropical homology.

Combinatorial motivation

$\mathcal{A} \subset \mathbb{R}^d$: real hyperplane arrangement.

Theorem (Zaslavsky '75)

The number of topes of $\mathcal{A}$ equals $(-1)^d \chi_{\mathcal{A}}(-1)$.



$\mathcal{M}(\mathcal{A}) := \mathbb{R}^d \setminus \mathcal{A}$, $\mathcal{A}^{\mathbb{C}} \subset \mathbb{C}^d$: compexification of $\mathcal{A}$.
Topological formulation: $\dim H_0(\mathcal{M}(\mathcal{A}); R) = \sum \dim H_i(\mathcal{M}(\mathcal{A}^{\mathbb{C}}); R)$.

Filtrations of the Tope Space

Zaslavsky: $\dim H_0(\mathcal{M}(\mathcal{A}); R) = \sum \dim H_i(\mathcal{M}(\mathcal{A}^{\mathbb{C}}); R)$.

Find a *filtration* $H_0(\mathcal{M}(\mathcal{A}); R) \cong R[\mathcal{T}] \supset \mathcal{G}_1 \supset \ldots \supset \mathcal{G}_d \supset \{0\}$ such that:

- $\dim \mathcal{G}_i/\mathcal{G}_{i+1} = \dim H_i(\mathcal{M}(\mathcal{A}^{\mathbb{C}}); R)$, via
- *isomorphisms* $\mathrm{bv}_i : \mathcal{G}_i/\mathcal{G}_{i+1} \cong H_i(\mathcal{M}(\mathcal{A}^{\mathbb{C}}); R)$ ($\cong \mathrm{OS}_i(M; R)$).

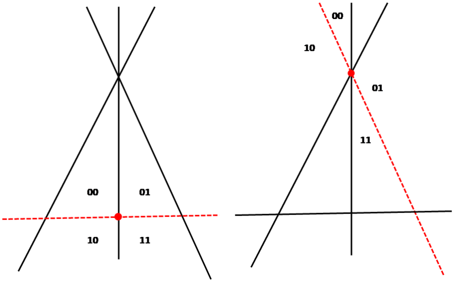
Three candidates:

1. Renaudineau–Shaw filtration ('23) $\mathcal{Q}_\bullet$;
2. Kalinin filtration ('05) $\mathcal{K}_\bullet$;
3. Varchenko–Gelfand dual degree filtration ('87) $\mathcal{P}_\bullet$.

Renaudineau–Shaw Filtration

- V: Vector space over $\mathbb{Z}/2\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})[V]$: group ring of V;
- $\mathcal{Q}_1(V) =$ augmentation ideal $= \{\sum w_v$ with even size support$\}$;
- $\mathcal{Q}_i(V) := \mathcal{Q}_1^i(V) = \{\gamma_1 \ldots \gamma_i : \gamma_j \in \mathcal{Q}_1(V)\}$. [Quillen '68]

Fix a complete flag of flats $\mathcal{F}$, $\mathcal{T}_{\mathcal{F}}$: topes that are *adjacent* to $\mathcal{F}$:

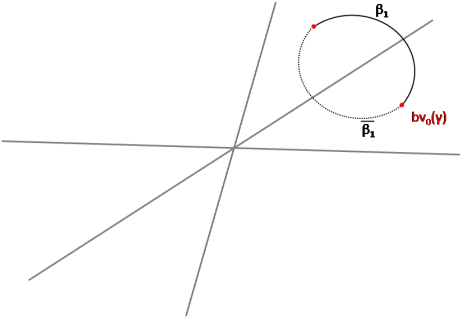


- $\mathcal{Q}_i(M) := \sum_{\mathcal{F}} \mathcal{Q}_i(\mathcal{T}_{\mathcal{F}})$.
- $\mathrm{bv}_i^{\mathrm{trop}} : \mathcal{Q}_i/\mathcal{Q}_{i+1} \to \mathrm{OS}_i(M; \mathbb{Z}/2\mathbb{Z})$ is given by $[(w_0 + w_{\epsilon_1}) \ldots (w_0 + w_{\epsilon_i})] \mapsto \epsilon_1 \wedge \ldots \wedge \epsilon_i$.

Kalinin Filtration

Everything with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

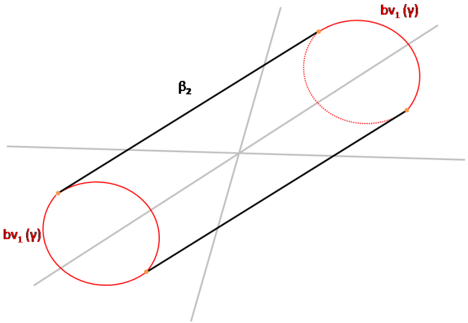
- Starting with the inclusion $\iota : \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{A}^{\mathbb{C}})$.
- $\mathrm{bv}_0 := \iota_* : H_0(\mathcal{M}(\mathcal{A})) \to H_0(\mathcal{M}(\mathcal{A}^{\mathbb{C}}))$.
- $\mathcal{K}_1 := \ker \mathrm{bv}_0$, $\gamma \in \mathcal{K}_1 \Rightarrow \partial\beta_1 = \gamma$.



- $\mathrm{bv}_1 : \mathcal{K}_1 \to H_1(\mathcal{M}(\mathcal{A}^{\mathbb{C}})), \gamma \mapsto \beta_1 + \overline{\beta_1}$.

Kalinin Filtration

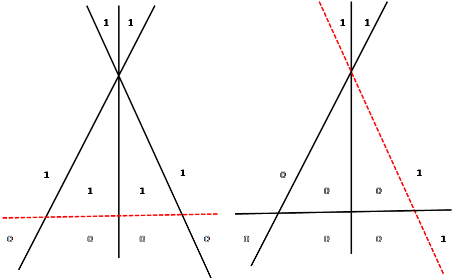
- $\mathcal{K}_2 := \ker \mathrm{bv}_1$, $\gamma \in \mathcal{K}_2 \Rightarrow \partial\beta_2 = \mathrm{bv}_1(\gamma)$.



- $\mathrm{bv}_2 : \mathcal{K}_2 \to H_2(\mathcal{M}(\mathcal{A}^{\mathbb{C}})), \gamma \mapsto \beta_2 + \overline{\beta_2}$, etc.
- $H_0(\mathcal{M}(\mathcal{A})) \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \dots$

With $\mathbb{Z}$ coefficients!

$H^0(\mathcal{M}(\mathcal{A});\mathbb{Z})$ is generated by *heaviside functions* h_e's as a ring:



Filtrate H^0 by $\mathcal{P}^i = \{\text{polynomials in } h_e \text{ of degree} \leq i\}$.

Take $\mathcal{P}_i = (\mathcal{P}^{i-1})^\perp = \{\gamma : h(\gamma) = 0, \forall h \in \mathcal{P}^{i-1}\}$.
$\text{bv}_i^{\mathcal{P}} : \mathcal{P}_i/\mathcal{P}_{i+1} \cong H_i(\mathcal{M}(\mathcal{A}^{\mathbb{C}});\mathbb{Z})$ given by [V–G '87] and [Denham '02].

Main Theorems

	Topo meaning	Easy to compute	Functorial	coeff.
$\mathcal{Q}_\bullet$	?	✓	✓	$\mathbb{Z}/2\mathbb{Z}$
$\mathcal{K}_\bullet$	✓	?	?	$\mathbb{Z}/2\mathbb{Z}$
$\mathcal{P}_\bullet$	✓-ish	✓	?	$\mathbb{Z}$

Theorem (Shaw–Y. 2024+)

- *With $\mathbb{Z}/2\mathbb{Z}$ coefficient, $\mathcal{K}_\bullet = \mathcal{Q}_\bullet = \mathcal{P}_\bullet$.*
- $\mathrm{bv}_i = \mathrm{bv}_i^{\mathcal{P}} \cong \mathrm{bv}_i^{\mathrm{trop}}$: *using $BZ : H^* \xrightarrow{\cong} OS^*$ from [Björner–Ziegler '92]*

$$\begin{array}{ccc} \mathcal{K}_i/\mathcal{K}_{i+1} & \xrightarrow{\mathrm{bv}_i = \mathrm{bv}_i^{\mathcal{P}}} & H_i(\mathcal{M}(\mathcal{A}^{\mathbb{C}}); \mathbb{Z}/2\mathbb{Z}) \\ \| \downarrow & & BZ^\vee \downarrow \\ \mathcal{Q}_i/\mathcal{Q}_{i+1} & \xrightarrow{\mathrm{bv}_i^{\mathrm{trop}}} & \mathrm{OS}_i(M; \mathbb{Z}/2\mathbb{Z}) \end{array}$$

Future Directions

- Upgrade the (Rau–)Renaudineau–Shaw theory to $\mathbb{Z}$-coefficients.
- Globalize other Varchenko–Gelfand constructions.

Thank you!