White's Conjecture for Paving Matroids

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Exchange Properties of Matroids

Definition

A *matroid* on a ground set E is a collection $\mathcal{B}$ of *bases* $B \subset E$ such that:

- (B1) $\mathcal{B} \neq \emptyset$
- (B2) $B \neq B', x \in B \setminus B' \Rightarrow \exists y \in B' \setminus B, B - x + y \in \mathcal{B}$

Theorem (Symmetric Exchange Property)

(B1) + (B2) are equivalent to (B1) +

- (B2') $B \neq B', x \in B \setminus B' \Rightarrow \exists y \in B' \setminus B, B - x + y, B' - y + x \in \mathcal{B}$

$$\begin{array}{ccc} B & \overset{x \qquad y}{\longleftarrow\!\longrightarrow} & B - x + y \\ B' & \overset{y \qquad x}{\longleftarrow\!\longrightarrow} & B' - y + x \end{array}$$

There is a long list of open questions on understanding the structure of bases of matroids. These conjectures suggest that matroids may possess much stronger structural properties than are currently known.

Bérczi–Jánosik–Mátravölgyi (2026)

White's Conjecture

Definition

Two tuples $(B_1, \ldots, B_n), (B'_1, \ldots, B'_n)$ of bases are *White connected* if one can be transformed to the other by sym. basis exchange one pair at a time.

$$(B_1, \ldots, \quad B_i \quad, \ldots, \quad B_j \quad, \ldots, B_n)$$
$$\Updownarrow$$
$$(B_1, \ldots, B_i - x + y, \ldots, B_j - y + x, \ldots, B_n)$$

Conjecture (White 1980)

Two tuples of bases are White connected if and only if their multi-set unions are equal.

Conjecture (Weak White's conjecture)

Allow replacing (B_i, B_j) by (B''_i, B''_j) with the same multi-set union directly.

Previous Works

White's conjecture is true for:

- Graphic matroids (Blasiak 2008)
- Matroids of rank ≤ 3 (Kashiwabara 2010)
- Sparse paving matroids (Bonin 2013)
- Strongly base-orderable matroids (Lasoń–Michałek 2014)

White's conjecture with $n = 2$ is true for:

- Split matroids (Bérczi–Schwarcz 2024)
- Regular matroids (Bérczi–Mátravölgyi–Schwarcz 2024)

White's conjecture is false (Larson 3rd July, 2026) :s

White's conjecture implies that the *toric ideal of a matroid* is generated by *symmetric exchange binomials*.

- Weak White's conjecture implies that it is generated by quadrics.

Main Theorem

A matroid is *paving* if every subset of size $\leq r-1$ is independent.

Theorem (Yu–Y. 2025+)

White's conjecture holds for paving matroids.

Why paving matroid?

- Conjecture (Mayhew–Newman–Welsh–Whittle): Asymptotically almost all matroids are paving.
- Theorem (Aliabadi 2026+): The number of paving matroids that are not sparse paving is logarithmic large.

$$p_n - sp_n \geq sp_{n,\lfloor n/2 \rfloor}^{1-o(1)}$$

- All matroids from Steiner systems are paving.
- Characterization using matroid polytopes and subdivisions.

Relaxation of Matroids

A subset $F \subset E$ is *stressed* if $M|_F$ and M/F are uniform.
Relaxation: Include extra bases to M by "perturbing" some stressed F.

Precise definition: Declare $S \subset E$ of size r to be a basis if $|S \cap F| > r(F)$.
In case F is a hyperplane, simply include all r-subsets of F.

"If M is X, then it is related to $U_{r,E}$ by a sequence of Y relaxations":

Matroid Class	Relaxation of
Sparse paving	Circuit-hyperplanes
Paving	Stressed hyperplanes
Split	Stressed subsets

Inductive Approach

Theorem (Han–Michałek–Weigert 2025+)

White's conj. holds for M iff it holds for M relaxing a circuit-hyperplane.

Corollary (Alternative Proof of [Bonin 2013])

White's conjecture holds for sparse paving matroids.

Theorem (Yu–Y. 2025+)

Let $\widetilde{M}$ be the relaxation of M at a stressed hyperplane H.

1. *If White's conj. holds for M, then White's conj. holds for $\widetilde{M}$;*
2. *If White's conj. holds for $\widetilde{M}$ and (*) holds for M, then White's conj. holds for M.*

(*): Any 2-step exchange sequence (of bases of M) in $\widetilde{M}$ can be replaced by an exchange sequence in M.

$$\mathbb{B} \to \widetilde{\mathbb{B}} \to \mathbb{B}' \quad \Rightarrow \quad \mathbb{B} \to \mathbb{B}_1 \to \ldots \to \mathbb{B}_\ell \to \mathbb{B}'$$

Proof of the Inductive Theorem

We consider three types of subsets $S \subset E$ of size r:

- $|S \setminus H| = 0$: Basis of $\widetilde{M}$ but not M
- $|S \setminus H| = 1$: Basis of both
- $|S \setminus H| \geq 2$: Basis of both or neither

Lemma

If X is of type 0 and Y is a basis of type ≥ 2, then they can be exchanged into bases of M.

Sketch of Proof of the Inductive Thm.

"$M \Rightarrow \widetilde{M}$": For both tuples, exchange till only type ≥ 1 (resp. type ≤ 1) subsets left.

"$\widetilde{M} \rightarrow M$": Start with an exchange sequence in $\widetilde{M}$, insert/replace steps using Lemma or $(*)$, keep track of monovariants to guarantee termination.

(*) for Paving Matroids

Need to resolve

$$\begin{array}{ccccccc} X & \overset{a \quad s}{\longleftrightarrow} & X'' \subset H & \overset{t \quad b}{\longleftrightarrow} & X' \\ Y & \overset{s \quad a}{\longleftrightarrow} & Y'' & \overset{b \quad t}{\longleftrightarrow} & Y' \end{array} \quad \text{or} \quad \begin{array}{ccccc} X & \overset{a \quad s}{\longleftrightarrow} & X'' \subset H & \overset{t \quad b}{\longleftrightarrow} & X' \\ Y & \overset{s \quad a}{\longleftrightarrow} & Y' & & Y' \\ Z & & Z & \overset{b \quad t}{\longleftrightarrow} & Z' \end{array}$$

Degree 2:

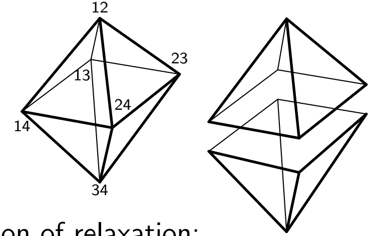
1. WLOG $X \cup Y = E, X \cap Y = \emptyset$.
2. WLOG $Y - b + a, Y - s + t, X - t + b$ are not bases hence span $H_{Y,as}, H_{Y,bt}, H_{X,ab}$ together with $H_{X,st} = H$.
3. WLOG $E = H_{X,st} \cup H_{X,ab} = H_{Y,as} \cup H_{Y,bt}$ or else any $p \in E$ not covered can be used for connection.
4. Now $\sum |H_{\cdot,\cdot} \cap H_{\cdot,\cdot}| = 6r - 8$ by the covering assumption, but it should be $\leq 6r - 12$ as each $|H_{\cdot,\cdot} \cap H_{\cdot,\cdot}| \leq r - 2$.

Degree 3: Similar approach but with even more cases.

Matroid Polytopes and Their Subdivisions

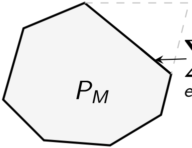
Definition

The *matroid polytope* of M is the convex hull of $\mathbf{e}_B := \sum_{i \in B} \mathbf{e}_i$'s in $\mathbb{R}^E$.
We can talk about *matroid subdivisions*.



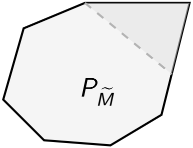
Polyhedral interpretation of relaxation:

Before relaxation



$$\sum_{e \in F} x_e \leq \mathrm{rank}(F)$$

After relaxation



Thank you & Diolch!